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Emergent CPT violation from the splitting of Fermi points

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In a fermionic quantum vacuum, the parameters k_μ of a CPT-violating Chern–Simons-like action term induced by CPT-violating parameters of the fermionic sector depend on the universality class of the system. As a concrete example, we consider the Dirac Hamiltonian of a massive fermionic quasiparticle and add a particular term with purely-spacelike CPT-violating parameters $b_\mu = (0, \mathbf{b})$. A quantum phase transition separates two phases, one with a fully-gapped fermion spectrum and the other with topologically-protected Fermi points (gap nodes). The emergent Chern–Simons "vector" $k_\mu = (0, \mathbf{k})$ now consists of two parts. The regular part, $\mathbf{k}^{\text{reg}}$, is an analytic function of $|\mathbf{b}|$ across the quantum phase transition and may be nonzero due to explicit CPT violation at the fundamental level. The anomalous (nonanalytic) part, $\mathbf{k}^{\text{anom}}$, comes solely from the Fermi points and is proportional to their splitting. In the context of condensed-matter physics, the quantum phase transition may occur in the region of the BEC–BCS crossover for Cooper pairing in the p–wave channel. For elementary particle physics, the splitting of Fermi points may lead to neutrino oscillations, even if the total electromagnetic Chern–Simons-like term cancels out.

Keywords: Lorentz noninvariance; superfluidity; quantum phase transition.

1. Introduction

Condensed-matter physics provides us with a broad class of quantum field theories which are not restricted by Lorentz invariance. This allows us to consider many problems in the relativistic quantum field theory of the electroweak Standard Model[1] from a more general perspective. Indeed, general consideration of quantum field theories have revealed that their basic properties are determined by momentum-space topology, which classifies the vacua of quantum field theory according to three universality classes.

In the present article, we apply the general topological argument to the particular case of a CPT-violating Chern–Simons-like term. For this case, certain calculations

in the framework of relativistic quantum field theory have turned out to be problematic because of an ambiguity tracing back to different regularization schemes. We find, on the other hand, that the Chern–Simons parameter can be expressed purely by a topological invariant in momentum space. In fact, this approach provides a topological regularization scheme which is more general than other schemes based on symmetry.

The topological approach holds for two common applications of relativistic quantum field theory, namely as a description of fundamental physics or as an emergent phenomenon of a fermionic quantum vacuum with the known elementary particles corresponding to the quasiparticles of the system. Since the topological approach has been proven correct in the class of condensed-matter systems with emerging relativistic quantum field theory in the low-energy corner, we expect that it may be relevant to Standard Model physics as well.

The momentum-space topology of the fermionic spectrum determines the universality classes of all known quantum vacua in which momentum is a well defined quantity (i.e., which have translation invariance); cf. Chapters 7 and 8 of Ref. 2. But the vacuum of the Standard Model above the electroweak transition (with vanishing fermion masses) is marginal and, strictly speaking, belongs to all three universality classes. The Standard Model vacuum above the electroweak transition has, in fact, a multiply degenerate Fermi point $\mathbf{p} = 0$ and the total topological charge of this point is zero. The implication is that momentum-space topology cannot protect this vacuum against decay into other (topologically-stable) vacua.

The Standard Model vacuum above the electroweak transition is only protected by symmetries, namely the continuous electroweak symmetry, the discrete symmetry discussed in Section 12.3.2 of Ref. 2, and the CPT symmetry. Explicit violation or spontaneous breaking of these symmetries (CPT in particular) leads to three possible scenarios, corresponding to the three universality classes of fermionic vacua:

(i) *Annihilation of Fermi points*— Fermi points with opposite topological charge merge and disappear. [Fermi points (gap nodes) $\mathbf{p}_a$ are points in three-dimensional momentum space at which the energy spectrum $E(\mathbf{p})$ of the fermionic quasiparticle has a zero, i.e., $E(\mathbf{p}_a) = 0$. Some early references on Fermi points include Refs. 3, 4, 5 and 6.] The annihilation of Fermi points allows the fermions to become massive, as happens with the quarks and charged leptons of the Standard Model below the electroweak transition.

(ii) *Splitting of Fermi points*— Fermi points split and separate along a spatial direction in four-dimensional energy-momentum space. The topological charges of the new isolated Fermi points are nonzero and the corresponding vacua are topologically protected. For Standard Model physics, the splitting of Fermi points gives rise to a CPT-violating Abelian Chern–Simons-like term[7,8]

$$S_{\text{CS-like}} = \int d^4x \; k_\mu \, \epsilon^{\mu\nu\rho\sigma} A_\nu(x) \, \partial_\rho A_\sigma(x) \; , \tag{1}$$

with gauge field $A_\mu(x)$, Levi–Civita symbol $\epsilon^{\mu\nu\rho\sigma}$, and a purely spacelike "vec-

tor" $k_\mu = (0, \mathbf{k})$. As will be explained later, the Fermi-point origin of such a Chern–Simons-like term is, in a way, complementary to the mechanism of the CPT anomaly[9,10,11,12] in relativistic quantum field theory. For condensed-matter physics, the prime example is superfluid ^{3}He–A; cf. Refs. 13 and 14. Here, the splitting is extremely large, with Fermi points separated by a "Planck-scale" momentum. But the splitting can, in principle, be controlled by a tunable interaction in the p–wave Cooper channel (see below).

(iii) *Formation of Fermi surfaces*— Fermi points split along the energy axis in four-dimensional energy-momentum space and give rise to a Fermi surface in three-dimensional momentum space. [Fermi surfaces S_a are two-dimensional surfaces in three-dimensional momentum space on which the energy spectrum $E(\mathbf{p})$ is zero, i.e., $E(\mathbf{p}) = 0$ for all $\mathbf{p} \in S_a$.] For Standard Model physics, this scenario would lead to a Chern–Simons-like term (1) with a purely timelike "vector" $k_\mu = (k_0, \mathbf{0})$. For condensed-matter physics, this scenario need not be prohibited by the helical instability of the vacuum, since the total k_0 obtained by summation over all Fermi surfaces can be zero due to the remaining symmetries of the fermions.[15]

The choice between these three scenarios depends on which symmetry-breaking mechanism is stronger. The important point is that there are only three possibilities corresponding to the three universality classes of fermionic vacua: (i) vacua with fully-gapped fermionic excitations; (ii) vacua with fermionic excitations characterized by Fermi points (these excitations behave as Weyl fermions close to the Fermi points); (iii) vacua with fermionic excitations characterized by Fermi surfaces.

Experimentally, the quantum phase transition between vacua of different universality classes can be investigated with laser-manipulated fermionic gases. The idea is to probe the crossover between the weak-coupling state described by Bardeen–Cooper–Schrieffer (BCS) theory and the strong-coupling limit described by Bose–Einstein condensation (BEC) of fermionic atom pairs. If the pairing occurs in the p–wave state, the BEC–BCS crossover may be accompanied by a quantum phase transition between vacua with Fermi points and fully-gapped vacua. Starting from the BEC phase, a marginal Fermi point is formed at the quantum phase transition, which splits into topologically-stable Fermi points as the interaction strength is reduced further (BCS phase).

The rest of the article elaborates these points. Section 2 explains the possible origin of CPT violation as a quantum phase transition which splits a multiply degenerate Fermi point and Section 3 discusses the induced Chern–Simons-like term with a purely spacelike vector. (Appendix A gives the Chern–Simons vector in the form of a momentum-space topological invariant.) Section 4 describes this action term in the context of superfluid ^{3}He–A. Section 5 considers the role of Fermi points for p–wave superconductors and the Standard Model (the possibility of a new mechanism for neutrino oscillations is also pointed out). Section 6 discusses the origin of a different type of Chern–Simons-like term with a purely timelike vector, as being due to the appearance of a Fermi surface (another possible consequence

being again neutrino oscillations). Section 7 mentions two other mechanisms for the splitting of Fermi points (or appearance of Fermi surfaces), namely finite-size effects and the presence of defects. Section 8 presents some concluding remarks and briefly discusses experiments of the BEC–BCS crossover.

2. CPT Violation as Quantum Phase Transition

As a typical example of the spacelike splitting of multiply degenerate Fermi points, we consider the marginal Fermi point of the Standard Model above the electroweak transition. In this section and the next, natural units are employed with $c = \hbar = 1$.

Take, for simplicity, a single pair of relativistic chiral fermions, that is, one right-handed fermion and one left-handed fermion. These are Weyl fermions with Hamiltonians $H_{\rm right} = \vec{\sigma}\cdot\mathbf{p}$ and $H_{\rm left} = -\vec{\sigma}\cdot\mathbf{p}$, where $\vec{\sigma}$ denotes the triplet of Pauli matrices. Each of these Hamiltonians has a topologically-stable Fermi point $\mathbf{p} = 0$, where the Hamiltonian is zero.

For a general system, relativistic or nonrelativistic, the stability of the a-th Fermi point is guaranteed by the topological invariant N_a, which can be written as a surface integral in energy-momentum space. In terms of the fermionic propagator $G(ip_0, \mathbf{p})$, the topological invariant is[2]

$$N_a = \frac{1}{24\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\;\mathrm{tr}\oint_{\Sigma_a} dS^\sigma\; G\frac{\partial}{\partial p_\mu}G^{-1}\; G\frac{\partial}{\partial p_\nu}G^{-1}\; G\frac{\partial}{\partial p_\rho}G^{-1}\;, \tag{2}$$

where Σ_a is a three-dimensional surface around the isolated Fermi point $p_{\mu a} = (0, \mathbf{p}_a)$ in four-dimensional energy-momentum space and 'tr' stands for the trace over the spin indices.

In our case, we have two Weyl fermions, $a = 1$ for the right-handed fermion and $a = 2$ for the left-handed one. The corresponding Green's functions are given by

$$G^{-1}_{\rm right}(ip_0, \mathbf{p}) = ip_0 - \vec{\sigma}\cdot\mathbf{p}\;,\quad G^{-1}_{\rm left}(ip_0, \mathbf{p}) = ip_0 + \vec{\sigma}\cdot\mathbf{p}\;. \tag{3}$$

The positions of the Fermi points coincide, $\mathbf{p}_1 = \mathbf{p}_2 = 0$, but their topological charges (2) are different, $N_1 = +1$ and $N_2 = -1$. For this simple case, the topological charge equals the chirality of the fermions, $N_a = C_a$.

The splitting of coinciding Fermi points can be described by the Hamiltonians $H_{\rm right} = \vec{\sigma}\cdot(\mathbf{p}-\mathbf{p}_1)$ and $H_{\rm left} = -\vec{\sigma}\cdot(\mathbf{p}-\mathbf{p}_2)$, with $\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{b}$ from momentum conservation. The real vector $\mathbf{b}$ is assumed to be odd under CPT, which introduces CPT violation into these Hamiltonians. The 4×4 matrix of the combined Green's function has the form

$$G^{-1}(ip_0, \mathbf{p}) = \begin{pmatrix} ip_0 - \vec{\sigma}\cdot(\mathbf{p}-\mathbf{b}) & 0 \\ 0 & ip_0 + \vec{\sigma}\cdot(\mathbf{p}+\mathbf{b}) \end{pmatrix}\;. \tag{4}$$

From Eq. (2) follows that $\mathbf{p}_1 = \mathbf{b}$ is the Fermi point with topological charge $N_1 = +1$ and $\mathbf{p}_2 = -\mathbf{b}$ the Fermi point with topological charge $N_2 = -1$.

Physically, it is perhaps more instructive to discuss the appearance and splitting of Fermi points by starting from a Dirac fermion (fermionic quasiparticle) with rest

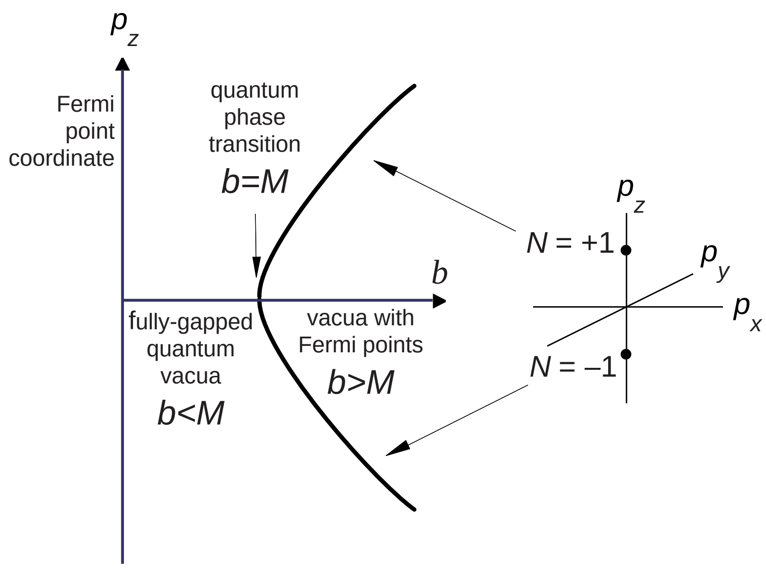

Fig. 1. Quantum phase transition at $b \equiv |\mathbf{b}| = M$ between anomaly-free, fully-gapped vacua for $b < M$ and vacua with topologically-protected Fermi points for $b > M$, where $\mathbf{b}$ is a CPT-odd parameter of the modified Dirac Hamiltonian (5). The two Fermi points are characterized by topological invariants $N = \pm 1$, as indicated by the inset on the right (the unit vector $\widehat{\mathbf{z}}$ in momentum space is parallel to $\mathbf{b}$). These Fermi points give an anomalous (nonanalytic) contribution to the induced Chern-Simons parameter $\mathbf{k}$.

energy M. Consider, then, a single Dirac fermion in the presence of a CPT-violating vector $\mathbf{b}$, with a modified Hamiltonian ($c = 1$)

$$H = \begin{pmatrix} \vec{\sigma}\cdot(\mathbf{p}-\mathbf{b}) & M \\ M & -\vec{\sigma}\cdot(\mathbf{p}+\mathbf{b}) \end{pmatrix} = H_{\mathrm{Dirac}} - \mathbb{1}_2 \otimes (\vec{\sigma}\cdot\mathbf{b}) \ . \tag{5}$$

This is the typical starting point for investigations of the effects of CPT violation in the fermionic sector; see, e.g., Refs. 16, 17, and 18. The energy spectrum of the Hamiltonian (5) is

$$E_\pm^2 = M^2 + |\mathbf{p}|^2 + b^2 \pm 2\sqrt{b^2 M^2 + (\mathbf{p}\cdot\mathbf{b})^2}\ , \tag{6}$$

with $b \equiv |\mathbf{b}|$.

Allowing for variable b in Eqs. (5) and (6), one finds a quantum phase transition at $b = M$ between fully-gapped vacua for $b < M$ and vacua with two Fermi points for $b > M$. These Fermi points are given by

$$\mathbf{p}_1 = \mathbf{b}\sqrt{1 - M^2/b^2}\ , \qquad \mathbf{p}_2 = -\mathbf{b}\sqrt{1 - M^2/b^2}\ . \tag{7}$$

From Eq. (2), now with a trace over the indices of the 4×4 Dirac matrices, follows that $\mathbf{p}_1$ is the Fermi point with topological charge $N_1 = +1$ and $\mathbf{p}_2$ the Fermi point with topological charge $N_2 = -1$ (see Fig. 1). The magnitude of the splitting of the two Fermi points is $2\sqrt{b^2 - M^2}$. At the quantum phase transition between

the vacua of the two universality classes (i.e., at $b = M$), the Fermi points with opposite charges annihilate each other and form a marginal Fermi point $\mathbf{p}_0 = 0$. The momentum-space topology of this marginal Fermi point is trivial (topological invariant $N_0 = +1 - 1 = 0$).

3. Emergent Chern–Simons-like Term

Let us now consider how the splitting of Fermi points in the fermionic sector induces an anomalous action term in the gauge-field sector. Start from the spectrum of a single electrically charged Dirac fermion (charge q) and again set $c = \hbar = 1$. In the presence of the vector potential $\mathbf{A}$ of a $U(1)$ gauge field, the minimally-coupled Hamiltonian is

$$H = \begin{pmatrix} \vec{\sigma}\cdot(\mathbf{p} - q\mathbf{A} - \mathbf{b}) & M \\ M & -\vec{\sigma}\cdot(\mathbf{p} - q\mathbf{A} + \mathbf{b}) \end{pmatrix} . \tag{8}$$

The positions of the Fermi points for $b \equiv |\mathbf{b}| > M$ are shifted due to the gauge field,

$$\mathbf{p}_a = q\mathbf{A} \pm \mathbf{b}\,\sqrt{1 - M^2/b^2}\ , \tag{9}$$

with a plus sign for $a = 1$ and a minus sign for $a = 2$. This result follows immediately from Eq. (7) by the minimal substitution $\mathbf{p}_a \to \mathbf{p}_a - q\mathbf{A}$, consistent with the gauge principle.

At this moment, we can simply use the general expression for the Wess–Zumino–Novikov–Witten (WZNW) action induced by Fermi points, which holds for any system. The WZNW action is represented by the following sum over Fermi points [see, e.g., Eq. (6a) in Ref. 19]:

$$S_{\mathrm{WZNW}} = -\frac{1}{12\pi^2} \sum_a N_a \int d^3x\ dt\ d\tau\ \mathbf{p}_a \cdot (\partial_t \mathbf{p}_a \times \partial_\tau \mathbf{p}_a)\ , \tag{10}$$

where $\tau \in [0, 1]$ is an additional coordinate which parametrizes a disc, with the usual spacetime at the boundary $\tau = 1$. The normalization factor $1/12\pi^2$ of this anomalous action is determined by a topological argument similar to that in Ref. 20; see Eqs. (3.10) and (3.11) of Ref. 21.

For relativistic quantum field theory and with different charges q_a at the different Fermi points, the dependence on $\mathbf{A}$ of the expression (10) comes from the shifts of the Fermi points, $\mathbf{p}_a = \mathbf{p}_a^{(0)} + q_a(\tau)\,\mathbf{A}$. Here, a parametrization is used in which the charges $q_a(\tau)$ are zero at the center of the disc, $q_a(0) = 0$, and equal to the physical charges at the boundary of the disc, $q_a(1) = q_a$. From Eq. (10), one obtains the general form for the Abelian Chern–Simons-like term

$$S_{\mathrm{CS-like}} = \frac{1}{24\pi^2} \sum_a N_a\, q_a^2 \int d^3x\, dt\ \mathbf{p}_a^{(0)} \cdot (\mathbf{A} \times \partial_t \mathbf{A})\ . \tag{11}$$

This result has the "relativistic" form (1) with a purely spacelike "vector"

$$k_\mu = (0, \mathbf{k}) = \left(0,\ (24\pi^2)^{-1} \sum_a \mathbf{p}_a^{(0)}\, q_a^2\, N_a\right)\ . \tag{12}$$

Note that only gauge invariance has been assumed in the derivation of Eq. (12). As shown in Appendix A, the Chern–Simons vector (12) can be written in the form of a momentum-space topological invariant.

Returning to the case of a single Dirac fermion with charge q and using Eqs. (12) and (7), one finds that the CPT-violating Chern–Simons parameter $\mathbf{k}$ can be expressed in terms of the CPT-violating parameter $\mathbf{b}$ of the fermionic sector,

$$\mathbf{k} = \frac{q^2}{12\pi^2}\,\theta(b-M)\,\mathbf{b}\,\sqrt{1-M^2/b^2}\;. \tag{13}$$

This contribution to $\mathbf{k}$ comes from the splitting of a marginal Fermi point and requires $b > M$, as indicated by the step function on the right-hand side $[\,\theta(x)=0$ for $x \le 0$ and $\theta(x)=1$ for $x>0\,]$.

In the context of relativistic quantum field theory, the existence of such a non-analytic contribution to $\mathbf{k}$ has also been found by Perez-Victoria[22] and Andrianov *et al.*[23] using standard regularization methods, but with a prefactor larger by a factor 3 and 3/2, respectively. The result (13), on the other hand, is determined by the general topological properties of the Fermi points and applies to nonrelativistic quantum field theory as well. In condensed-matter quantum field theory, the result was obtained without ambiguity, since the microphysics is known at all scales and the regularization occurs naturally.

The induced Chern–Simons vector (13) from explicit CPT violation in the Dirac Hamiltonian (8) differs from the anomalous result in chiral gauge theory over topologically nontrivial spacetime manifolds.[9,10,11,12] Most importantly, the result (13) holds for a vectorlike gauge theory (with a Dirac fermion), whereas the CPT anomaly appears exclusively in chiral gauge theories (with Weyl fermions). There are, however, also similarities. First, the result (13) arises only from a strong enough source of CPT violation ($b \equiv |\mathbf{b}| > M$) but can have an arbitrarily small mass scale ($2\sqrt{b^2-M^2}$). This behavior resembles that of the CPT anomaly in the usual form, which requires a large enough ultraviolet cutoff (Pauli–Villars mass or inverse lattice spacing) but can have an arbitrarily small mass scale ($1/L$, for size L of the compact dimension with periodic spin structure). Second, the pattern of two separated Fermi points is reminiscent of the different options for the definition of the fermion measure in the relevant chiral lattice gauge theory, where the choice of one or the other option leads to the CPT anomaly (see, in particular, Section 5.3 of Ref. 11).

4. WZNW Action for Superfluid ^{3}He–A

The Bogoliubov–Nambu Hamiltonian which qualitatively describes fermionic quasiparticles in ^{3}He–A is given by

$$H = \begin{pmatrix} p^2/2m_3-\mu & c_\perp\,\mathbf{p}\cdot(\widehat{\mathbf{e}}_1+i\,\widehat{\mathbf{e}}_2) \\ c_\perp\,\mathbf{p}\cdot(\widehat{\mathbf{e}}_1-i\,\widehat{\mathbf{e}}_2) & -p^2/2m_3+\mu \end{pmatrix}\;, \tag{14}$$

with the mass m_3 of the ^{3}He atom, the orthonormal triad $(\widehat{\mathbf{e}}_1,\,\widehat{\mathbf{e}}_2,\,\widehat{\mathbf{l}}=\widehat{\mathbf{e}}_1\times\widehat{\mathbf{e}}_2)$, and the maximum transverse speed $c_\perp$ of the quasiparticles. The unit vector $\widehat{\mathbf{l}}$

corresponds to the direction of the orbital momentum of the Cooper pairs. The energy spectrum of these Bogoliubov–Nambu fermions is

$$E^2 = \left(\frac{p^2}{2m_3} - \mu\right)^2 + c_\perp^2\, \left(\mathbf{p}\times\widehat{\mathbf{l}}\right)^2 \ . \tag{15}$$

This last expression for the energy makes clear that Fermi points (with $E = 0$) only occur for the case of positive chemical potential, $\mu > 0$. These Fermi points are $\mathbf{p}_1 = p_F\,\widehat{\mathbf{l}}$ and $\mathbf{p}_2 = -p_F\,\widehat{\mathbf{l}}$, with Fermi momentum $p_F = \sqrt{2\mu\, m_3}$. From the general expression (10), one finds the following Wess–Zumino–Novikov–Witten (WZNW) action for superfluid ^{3}He–A (cf. Refs. 14 and 19):

$$S^{\text{anomalous}}_{\text{WZNW}} = -\frac{\hbar K_0}{2}\int d^3x\ dt\ d\tau\, \widehat{\mathbf{l}}\cdot\left(\partial_t\widehat{\mathbf{l}}\times\partial_\tau\widehat{\mathbf{l}}\right)\ , \tag{16}$$

with coefficient $K_0 \equiv (p_F/\hbar)^3/3\pi^2$. This term has a nonanalytic dependence on μ through p_F and plays a crucial role in the orbital dynamics of ^{3}He–A.

The effective gauge field, which emerges for "relativistic" fermions in the vicinity of the Fermi points, is the collective mode of the shift of the Fermi points, $\mathbf{A}' = p_F\,\widehat{\delta\mathbf{l}}$, where $\widehat{\delta\mathbf{l}}$ is a perturbation around the vacuum value, $\widehat{\mathbf{l}} = \widehat{\mathbf{l}}_0 + \widehat{\delta\mathbf{l}}$. Expanding Eq. (16) in terms of $\mathbf{A}' \equiv \hbar\,\mathbf{A} = p_F\,\widehat{\delta\mathbf{l}}$, one finds that the effective Chern–Simons-like term in ^{3}He–A has the "relativistic" form (1) with spacelike "vector"

$$k_\mu = (0, \mathbf{k}) = \left(0,\ (12\pi^2)^{-1}\, p_F\, \widehat{\mathbf{l}}_0\right)\ . \tag{17}$$

As mentioned above, the vector $\widehat{\mathbf{l}}_0$ gives the direction of the orbital momentum of the Cooper pairs. The nonvanishing angular momentum (a T-odd vector) makes clear that time-reversal symmetry is spontaneously broken in the quantum vacuum of ^{3}He–A . (For relativistic quantum field theory which emerges in the vicinity of Fermi points, this implies T and CPT violation for the quasiparticles, e.g., the photon and the electron. Indeed, it is possible to construct a light clock which ticks differently if the velocities are reversed.[10])

For ^{3}He–A, there is also a regular part[14,19] of the WZNW action,

$$S^{\text{regular}}_{\text{WZNW}} = \int d^3x\ dt\ d\tau\ \frac{\hbar n}{2}\,\widehat{\mathbf{l}}\cdot\left(\partial_t\widehat{\mathbf{l}}\times\partial_\tau\widehat{\mathbf{l}}\right)\ , \tag{18}$$

with n the particle density of ^{3}He atoms. This regular part comes from the angular momentum of the liquid (angular momentum density $\mathbf{L} = \hbar\widehat{\mathbf{l}}\ n/2$), whereas the nonanalytic contribution (16) comes from the Fermi points and is proportional to their splitting ($2p_F\,\widehat{\mathbf{l}}$). (In bulk nonsuperconducting materials, there may be a similar nonanalytic contribution to the Hall conductivity; see Appendix A.) The regular contribution does not depend on the presence of Fermi points and remains when the Fermi points disappear by a quantum phase transition.

The disappearance of Fermi points takes place for strong interactions in the Cooper channel, when the quantum phase transition to the fully-gapped state occurs. Starting from the Fermi-point phase ($\mu > 0$) and increasing the interaction

strength, the value of μ drops and, at a particular value of the interaction strength, the momentum-space topology changes. According to Eq. (15), the two Fermi points annihilate each other at $\mu = 0$ and disappear for $\mu < 0$; cf. Sections 6.2 and 6.5 of Ref. 14. The quantum phase transition at $\mu = 0$ may be called a Lifshitz transition, by analogy with the quantum phase transitions in metals at which the momentum-space topology of the Fermi surface changes. Fermi points give rise to the chiral anomaly and to anomalous (nonanalytic) terms in the orbital dynamics. These anomalies are absent in the fully-gapped state ($\mu < 0$).

Real ^{3}He–A lives, however, in the $\mu > 0$ region of the phase diagram. A fully-gapped vacuum on the other side of the phase boundary (that is, a vacuum without Fermi points, but with the same breaking of time-reversal symmetry) can perhaps be obtained in laser-manipulated Fermi gases; see Section 8 for further remarks.

5. Fermi Points for *p*–Wave Superconductors and Standard Model

The vector $\mathbf{k}$ of the Abelian Chern–Simons-like term induced by all Fermi points is given by Eq. (12). Considering massless fermions with charges q_a and taking into account that the topological charge N_a for Weyl fermions coincides with their chirality $C_a = \pm 1$ (see Section 2), one obtains

$$\mathbf{k} = \frac{1}{24\pi^2} \sum_a \mathbf{p}_a \, q_a^2 \, C_a \ , \tag{19}$$

where the integer a labels the Fermi points $\mathbf{p}_a$.

For ^{3}He–A, the sum of Eq. (19) gives $\sum_a \mathbf{p}_a \, q_a^2 \, C_a = 2p_F \, \widehat{\mathbf{l}}_0$ and the induced vector $\mathbf{k}$ is nonzero. But, for other vacua, the induced vector $\mathbf{k}$ can be zero after summation. An example of this cancellation is provided by the so-called α–phase of spin-triplet pairing in superconductors; cf. Ref. 13. This phase contains eight Fermi points $\mathbf{p}_a$ $(a = 1, \ldots, 8)$ at the vertices of a cube in momentum space,[24] giving rise to four right-handed and four left-handed Weyl fermions. In terms of the Cartesian unit vectors $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$, the four Fermi points with right-handed Weyl fermions ($C_a = +1$, for $a = 1, \ldots, 4$) are given by

$$\begin{aligned} \mathbf{p}_1 &= \frac{p_F}{\sqrt{3}} \left(\widehat{\mathbf{x}} + \widehat{\mathbf{y}} + \widehat{\mathbf{z}} \right) \ , \quad \mathbf{p}_2 = \frac{p_F}{\sqrt{3}} \left(-\widehat{\mathbf{x}} - \widehat{\mathbf{y}} + \widehat{\mathbf{z}} \right) \ , \\ \mathbf{p}_3 &= \frac{p_F}{\sqrt{3}} \left(\widehat{\mathbf{x}} - \widehat{\mathbf{y}} - \widehat{\mathbf{z}} \right) \ , \quad \mathbf{p}_4 = \frac{p_F}{\sqrt{3}} \left(-\widehat{\mathbf{x}} + \widehat{\mathbf{y}} - \widehat{\mathbf{z}} \right) \ , \end{aligned} \tag{20}$$

while the four Fermi points with left-handed Weyl fermions ($C_a = -1$, for $a = 5, \ldots, 8$) have opposite vectors. The vector $\mathbf{k}$ from all Fermi points vanishes, since $q_a^2 = 1$ for the fermion charges $q_a = \pm 1$ and $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = 0$ for the tetrahedron (20). The regular part of $\mathbf{k}$ also vanishes because of the discrete cubic symmetry of the superconducting vacuum.

To the best of our knowledge, the α–phase in superconductors has not yet been established experimentally. The example is, however, useful in that it demonstrates that the total topological charge of the Fermi points is zero for condensed-matter

physics, $\sum_a N_a = 0$. For relativistic quantum field theory emerging near Fermi points, this implies an equal number of right- and left-handed fermions (some of which may have vanishing charges, though).

Consider, then, the $SU(3)_{\text{color}} \times SU(2)_L \times U(1)_Y$ Standard Model[1] of elementary particle physics above the electroweak transition (energy scale $E_{\text{weak}} \equiv G_{\text{Fermi}}^{-1/2}\,(\hbar c)^{3/2} \approx 300\,\text{GeV}$). For one family and with a right-handed neutrino included, the model contains eight Fermi points with $N = +1$ and eight Fermi points with $N = -1$, all located at $\mathbf{p} = 0$. The CPT-violating shifts $\mathbf{b}_a$ of the Fermi points may, in principle, be different for the different species of fermions ($a = 1, \ldots, 16$), but gauge invariance imposes certain restrictions; cf. Section 5 of Ref. 9. Recall the representations of the sixteen left- and right-handed fermions:

$$
\begin{aligned}
L &: \Big[\, (3,2)_{1/3} \,\Big]_{\text{quarks}} + \Big[\, (1,2)_{-1} \,\Big]_{\text{leptons}} \,, \\
R &: \Big[\, (3,1)_{4/3} + (3,1)_{-2/3} \,\Big]_{\text{quarks}} + \Big[\, (1,1)_{-2} + (1,1)_{0} \,\Big]_{\text{leptons}} \,,
\end{aligned}
\tag{21}
$$

where the entries in parentheses denote $SU(3)$ and $SU(2)$ representations and the suffix the value of the hypercharge Y. The electric charge Q is given by the combination $Y/2 + I_3$, with I_3 the weak isospin from the diagonal Hermitian generator T_3 of the $SU(2)$ Lie algebra. The specific values of the hypercharge and weak isospin are $Y_a \in \{-2, -1, -2/3, 0, +1/3, +4/3\}$ and $I_{3a} \in \{-1/2, 0, +1/2\}$.

For the $U(1)$ hypercharge gauge field of the Standard Model, the charge q_a in the sum (19) must be replaced by Y_a. It could then be that the values of $\mathbf{p}_a$ have a special pattern, similar to the pattern (20) of α–phase superconductors, so that the sum (19) vanishes exactly. A complete analysis of this problem lies outside the scope of the present article, but one simple solution can already be discussed.

Given the hypercharges Y_a of the Standard Model (21), the following pattern of Fermi points may be assumed:

$$
\mathbf{p}_a^{(f)} = Y_a \;\, \mathbf{p}^{(f)} \,, \tag{22}
$$

where, for the moment, the family index f is set equal to 1. The sixteen Fermi points from Eq. (22) for $f = 1$ depend on a single vector, $\mathbf{p}^{(1)}$. The factorized pattern (22) makes for a vanishing sum in Eq. (19) with q_a replaced by the hypercharge Y_a and also for the sum with q_a replaced by the weak isospin I_{3a}. Indeed, the very same sums occur for the perturbative gauge anomalies, which are known to cancel out (cf. the last two papers of Ref. 1).

With three known fermion families, $f = 1, 2, 3$, the pattern (22) is still a possibility. But, in general, the basic vectors $\mathbf{p}^{(f)}$ have different lengths,

$$
\left| \mathbf{p}^{(f)} \right| \neq \left| \mathbf{p}^{(f')} \right| \,, \quad \text{for} \quad f \neq f' \in \{1, 2, 3\} \,, \tag{23}
$$

and may or may not be aligned. The pattern (22)–(23), or some other condition on the $\mathbf{p}_a^{(f)}$ with the same effect, may indeed be required by the very tight experimental limit[7] on an electromagnetic Chern–Simons-like term, $|\,\mathbf{k}^{\text{exp}}| \lesssim 10^{-33}\,\text{eV}/c$.

Without cancellations, one would need to explain the extreme smallness of each value $|\mathbf{p}_a^{(f)}|$ separately.

For neutrinos, it is, in principle, possible that the splitting mechanism is stronger than the mechanism leading to mass formation (cf. Section 12.3.4 of Ref. 2) and that splitting occurs instead of mass generation. Or one has both, but with $|\mathbf{b}| > M$, so that the Fermi points survive (cf. Section 2). The typical energy scale associated with the $\mathbf{p}_a^{(f)}$ of the massless (or nearly massless) neutrinos is perhaps given by $(E_{\rm weak}/E_{\rm Planck})^2 \times E_{\rm Planck} \approx 10^{-5}\,\mathrm{eV}$, if the corresponding CPT violation is a "reentrance effect" from nonsymmetric physics at the fundamental scale $E_{\rm Planck} \equiv \sqrt{\hbar\, c^5/G}$, as discussed in Section 12.4.3 of Ref. 2.

The CPT-violating splittings (22)–(23) for the neutrinos have no direct consequences for the electromagnetic sector, since the neutrinos are electrically neutral. However, different splittings for the different species of neutrinos may cause neutrino oscillations. The basic idea is that there are three distinct propagation states for the left-handed neutrinos [cf. Eq. (6) for $M = 0$, with $\mathbf{b}$ replaced by the relevant $c\,\mathbf{p}_a^{(f)}$ and appropriate sign], which need not be the same as the three interaction states. For nontrivial mixing angles and large enough energy of the initial neutrino (with definite momentum and flavor f), the oscillation probabilities $P_{ff'}$ will be anisotropic but essentially energy independent; cf. Section III B of Ref. 25. Note that this Fermi-point-splitting mechanism for neutrino oscillations differs fundamentally from recent models based on CPT-violating neutrino masses (see, e.g., Ref. 26 and references therein).

6. Timelike Parameters and Fermi Surfaces

The Hamiltonian for a massive Dirac particle with an additional real CPT-violating timelike parameter b_0 is

$$H = \begin{pmatrix} \vec{\sigma}\cdot(c\,\mathbf{p}-\mathbf{b}) - b_0 & M \\ M & -\vec{\sigma}\cdot(c\,\mathbf{p}+\mathbf{b}) + b_0 \end{pmatrix} \,. \tag{24}$$

The discrete symmetries of this Hamiltonian have been studied, for example, in Ref. 18. For zero $\mathbf{b}$, the energy eigenvalues are given by

$$E_\pm^2 = M^2 + (c\,|\mathbf{p}| \pm b_0)^2 \,. \tag{25}$$

Topologically, this energy spectrum is similar to the one of Bogoliubov–Nambu fermions in s–wave superconductors where the role of the energy M is played by the gap Δ. The spectrum (25) is fully gapped for $M \neq 0$, but has a spherical Fermi surface of radius $|b_0|/c$ for $M = 0$. This Fermi surface $|\mathbf{p}| = |b_0|/c$ is marginal, as it consists of two identical Fermi surfaces with opposite global topological invariants from the left- and right-handed fermions. [The global topological invariant N_a for the Fermi surface S_a is defined by Eq. (2), with a three-dimensional surface Σ_a around S_a at $p_0 = 0$.] The total topological charge describing the Fermi surface $|\mathbf{p}| = |b_0|/c$ is thus zero and the Fermi surface is not protected by topology (indeed, it disappears for $M \neq 0$).

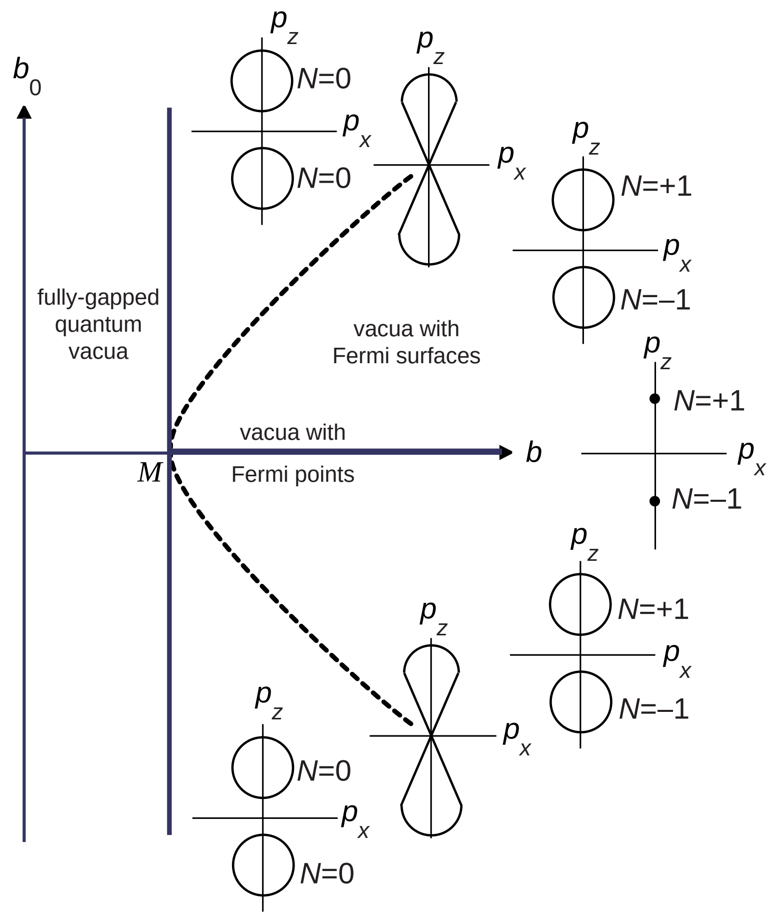

Fig. 2. Phase diagram for parameters b_0 and $b \equiv |\mathbf{b}|$ of the modified Dirac Hamiltonian (24). For $b < M$, there are fully-gapped quantum vacua. For $b \geq M$, there are vacua with Fermi surfaces (shown schematically by the insets, with p_y suppressed), except for an open line segment $b > M$ at $b_0 = 0$ with topologically-protected Fermi points (shown by the middle inset on the right) and a line $b = M$ with topologically-unprotected Fermi points (not shown by insets). The values of the global topological invariants N_a for the Fermi surfaces S_a or Fermi points $\mathbf{p}_a$ are indicated, where N_a is defined by Eq. (2) with Σ_a a three-dimensional surface around S_a or $\mathbf{p}_a$ at $p_0 = 0$. The dashed line $b^2 - b_0^2 = M^2$ (with Fermi surfaces meeting in $\mathbf{p} = \mathbf{0}$) marks a quantum phase transition where the global topological charges of the Fermi surfaces change (see main text).

Figure 2 clarifies the appearance of Fermi surfaces and Fermi points in a $(b_0, |\mathbf{b}|)$ phase diagram. The line $|\mathbf{b}|^2 - b_0^2 = M^2$ (with Fermi surfaces meeting in $\mathbf{p} = \mathbf{0}$) marks a quantum phase transition where the global topological charges N_a of the Fermi surfaces change. The Fermi surfaces of vacua with $|\mathbf{b}|^2 - b_0^2 > M^2$ inherit the topological charges $N = \pm 1$ from the Fermi points of the line segment $|\mathbf{b}| > M$ at $b_0 = 0$. The Fermi surfaces of vacua with $0 < |\mathbf{b}|^2 - M^2 < b_0^2$ have trivial global topology, $N = 0$, and shrink to points at $|\mathbf{b}| = M$. These topologically-unprotected

points with $N = 0$ then disappear for $|\mathbf{b}| < M$ (fully gapped vacua). The location of the quantum phase transition, $|\mathbf{b}|^2 - b_0^2 = M^2$, is Lorentz-invariant. But the parameters b_0 and $|\mathbf{b}|$ are determined in a preferred reference frame (the frame of the heat bath in the context of condensed-matter physics), which explains why other lines of the diagram change under boosts.

The analog of the Hamiltonian (24) with massless "relativistic" fermions, $M = 0$, occurs for ^{3}He–A; cf. Refs. 2 and 15. The flow of the superfluid component with respect to the heat bath produces effective chemical potentials μ_a, which are opposite for left- and right-handed fermions. After rescaling and deformation in momentum space, one has

$$H = \begin{pmatrix} \vec{\sigma}\cdot(c\,\mathbf{p} - \mathbf{b}) - \bar{\mu} & 0 \\ 0 & -\vec{\sigma}\cdot(c\,\mathbf{p} + \mathbf{b}) + \bar{\mu} \end{pmatrix} , \tag{26}$$

where $\bar{\mu} \equiv \mu_1 = -\mu_2 = p_F\,\mathbf{v}_s \cdot \widehat{\mathbf{l}}$, for a superfluid velocity $\mathbf{v}_s$ with respect to the heat bath. The two Fermi points $\mathbf{p}_a = \pm\mathbf{b}/c$ for the line segment $|\mathbf{b}| > 0$ at $\bar{\mu} = 0$ are transformed into two Fermi surfaces for $\bar{\mu} \neq 0$, given by $|c\,\mathbf{p} - \mathbf{b}| = |\bar{\mu}|$ and $|c\,\mathbf{p} + \mathbf{b}| = |\bar{\mu}|$. For $\mathbf{b} \neq \mathbf{0}$, these two Fermi surfaces cannot cancel each other, as they do not coincide.

According to Eq. (19.10) of Ref. 2, the k_0 contribution to the induced Chern–Simons-like term (1) has the Lagrange density

$$\mathcal{L}_{\mathrm{CS-like}} = -\frac{1}{8\pi^2}\,\mathbf{A}\cdot(\nabla\times\mathbf{A})\sum_a C_a\,q_a^2\,\mu_a/c\,. \tag{27}$$

For ^{3}He–A, this term is responsible for the observed helical instability of the "vacuum" in the presence of superflow.[2,13] In general, it is known that a Chern–Simons-like term with timelike vector k_μ, combined with the Maxwell term, leads to vacuum instability; cf. Refs. 7 and 8. [Note that the argument of Ref. 27 against a (timelike) induced Chern–Simons-like term from a fundamental CPT-violating quantum field theory is simply not applicable to the effective theory of the ^{3}He–A liquid, since its "vacuum" can really be unstable.]

The Hamiltonian (26) does not mix left- and right-handed fermions and is equivalent to Eq. (24) for $M = 0$, if we are only interested in the k_0 contribution to the Chern–Simons-like term (1). This implies that Eq. (27) should also be valid for massless relativistic fermions (the k_0 value determined by the Fermi surfaces does not depend on the regularization; cf. Ref. 28). Of course, there could be an additional regular part of $k_\mu = (k_0, 0, 0, 0)$, which is determined by CPT violation at the fundamental level and which does not depend on the presence of Fermi surfaces.

For the Standard Model fermions (21), there may be a pattern in the values of $b_{0a}^{(f)}$ of the Hamiltonian (24) for $\mathbf{b} = M = 0$, which nullifies the sum in Eq. (27) with μ_a replaced by $b_{0a}^{(f)}$ and q_a by Y_a or I_{3a}. As explained in the previous section, one example of such a pattern is

$$b_{0a}^{(f)} = Y_a\,b_0^{(f)}\,, \quad \left|b_0^{(f)}\right| \neq \left|b_0^{(f')}\right|\,, \quad \text{for} \quad f \neq f' \in \{1,2,3\}\,. \tag{28}$$

Different values of $b_{0a}^{(f)}$ for the different species of neutrinos may again cause neutrino oscillations [cf. Eq. (25) for $M = 0$, with b_0 replaced by the relevant $b_{0a}^{(f)}$ and appropriate sign].

7. Finite-Size Effects and Defects

In the context of relativistic quantum field theory (specifically, chiral gauge theory), an anomalous Chern–Simons-like term has been found to originate from nontrivial space topology.[9,10,11,12] For the present article, the splitting of Fermi points or the appearance of a Fermi surface (both of which induce a Chern–Simons vector $\mathbf{k}$) may also be caused by finite-size effects at the fundamental level or by defects in the fabric of space. If so, $\mathbf{b}$ would be either inversely proportional to the size of the compact dimension, $|\mathbf{b}| \propto 1/L$ (cf. Ref. 9), or depend on the distance from the space defect (cf. Ref. 12).

In condensed-matter physics, an interesting example is given by a combined topological defect in ^{3}He–A, namely vortex–disgyration. Its vortex part is characterized by the circulating superfluid velocity $\mathbf{v}_s(\mathbf{r}) = \hbar/(2m_3\,\rho)\,\widehat{\boldsymbol{\phi}}$, with ρ the distance from the vortex axis and m_3 the mass of the ^{3}He atom. The disgyration is represented by the azimuthal arrangement of the $\widehat{\mathbf{l}}$ field, $\widehat{\mathbf{l}}(\mathbf{r}) = \widehat{\boldsymbol{\phi}}$. This type of topological texture occurs in the asymptotic region of the so-called circular Mermin–Ho vortex; cf. Fig. 1a (right) of Ref. 29. The vortex produces the component $b_0 = p_F\,\mathbf{v}_s(\mathbf{r}) \cdot \widehat{\mathbf{l}}(\mathbf{r}) = \hbar\,p_F/(2m_3\,\rho)$ and the azimuthal disgyration gives the vector $\mathbf{b} = \hbar\,p_F/(2m_3\,\rho)\,\widehat{\boldsymbol{\phi}}$. This last vector induces a space-dependent Chern–Simons vector $\mathbf{k}(\mathbf{r})$ of the type discussed in Section II C of Ref. 12.

8. Conclusion

Timelike and spacelike CPT-violating parameters b_μ in the fermionic sector lead to different momentum-space topology and, hence, to a different response of the parameters k_μ of the Chern–Simons-like term in the effective gauge-field action. All three universality classes of fermionic vacua (with mass gaps, Fermi points, or Fermi surfaces) can play a role. The induced "vector" k_μ depends on the particular universality class chosen by the system. These observations are also relevant to elementary particle physics, both if the Standard Model is an emergent phenomenon of a fermionic quantum vacuum or if the Standard Model is part of a fundamental theory. Of particular interest may be a new CPT-violating mechanism for neutrino oscillations, as discussed in the last paragraphs of Sections 5 and 6.

A purely spacelike vector $k_\mu = (0, \mathbf{k})$ in the CPT-violating Chern–Simons-like action (1) consists of two parts. The regular part, $\mathbf{k}^{\text{reg}}$, can be nonzero due to explicit CPT violation at the fundamental level. For the concrete case of the Dirac Hamiltonian (8), $\mathbf{k}^{\text{reg}}$ is a regular function of the CPT-violating parameter $\mathbf{b}$. That is, $\mathbf{k}^{\text{reg}}$ is continuous across the quantum phase transition at $b \equiv |\mathbf{b}| = M$ between the vacua of the different universality classes. The anomalous (nonanalytic) part,

$\mathbf{k}^{\text{anom}}$, comes solely from the Fermi points (present for $b > M$) and is proportional to their splitting. This nonanalytic contribution is regularization independent. For the case of a single massive Dirac fermion, the vector $\mathbf{k}^{\text{anom}}$ is given by Eq. (13).

Moreover, the general regularization-independent expression (12) for $\mathbf{k}^{\text{anom}}$ depends only on the topological properties of the Fermi points (cf. Appendix A) and is applicable to any system which contains one or more pairs of Fermi points, even if the system does not obey relativistic invariance. The derivation of our expression used only gauge invariance. This is one of many examples where topological effects do not require the mathematics of relativistic invariance. Indeed, the result is applicable to nonrelativistic ^{3}He–A as well.

The example of ^{3}He–A demonstrates that there may be a regular part, $\mathbf{k}^{\text{reg}}$, which does not depend on the existence of Fermi points. This part comes from the angular momentum of the liquid (corresponding to spontaneously broken time-reversal symmetry) and is the same with or without Fermi points. In contrast, the nonanalytic part, $\mathbf{k}^{\text{anom}}$, comes from anomalies generated by the Fermi points. The total spacelike Chern–Simons-like term, with vector $\mathbf{k} \propto p_F\,\mathbf{l}$, is responsible for the orbital dynamics (i.e., the dynamics of the unit vector $\widehat{\mathbf{l}}$ directed along the orbital angular momentum of the Cooper pairs).

Real ^{3}He–A lives deep inside the Fermi-point region of the phase diagram where the regular and nonanalytic terms almost cancel each other. This last circumstance impacts on many phenomena of ^{3}He–A including the dynamics of topological defects, e.g., quantized vortices. There would be no Fermi points and related anomalies on the other side of the quantum phase transition.

This region of anomaly-free vacua, as well as the quantum phase transition between vacua with Fermi points and fully-gapped vacua, can perhaps be probed with laser-manipulated Fermi gases. Recently, the condensation of pairs of fermionic ^{40}K atoms has been reported[30] in the crossover regime (the so-called BEC–BCS crossover). In this experiment, a magnetic-field Feshbach resonance was used to control the interactions in the Cooper channel. But the system considered has (spin-singlet) s–wave pairing and there are fully-gapped vacua on both sides of the crossover, without quantum phase transition.

For the case of pairing interactions in the (spin-triplet) p–wave channel, the modulus b of the parameter $\mathbf{b}$ in the Hamiltonian (5) can be taken to be proportional to the inverse strength of the atom–atom interactions in the p–wave channel and its direction $\widehat{\mathbf{b}} \equiv \widehat{\mathbf{z}}$ to be along the unit vector $\widehat{\mathbf{l}}$ of the orbital angular momentum of the Cooper pairs. We may, then, expect the BEC–BCS crossover to be accompanied by the quantum phase transition of Fig. 1, with a change in the momentum-space topology of the vacuum and the universality class. Similar quantum phase transitions may be relevant to Standard Model physics.

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Appendix A. Momentum-Space Topological Invariant

In this appendix, we discuss the origin of the gauge invariance of the induced Chern–Simons-like term, even though it contains explicitly the gauge potential **A**. The Chern–Simons-like term (1) becomes gauge invariant if, for a special reason, the Chern–Simons vector **k** does not depend on space or time. There must then be a protection mechanism which keeps the vector **k** constant under perturbations of the system. Such a protection mechanism is well known in condensed-matter systems, where it has a topological origin. Certain physical parameters of these systems, such as the Hall and the spin-Hall conductivity, are robust to perturbations, because they can be expressed in terms of momentum-space invariants or more complicated momentum-space invariants protected by symmetry; cf. Refs. 2 and 31. The question, now, is if topological protection also operates for the anomalous contributions to the Chern–Simons vector **k** coming the Fermi points.

We start by rewriting the momentum-space topological invariant N_a, as defined by Eq. (2) in terms of the fermionic propagator $G(ip_0, \mathbf{p})$. For this purpose, we introduce a matrix density in four-dimensional energy-momentum space,

$$\mathcal{N}_\nu \equiv \frac{1}{24\pi^2}\, e_{\kappa\lambda\mu\nu}\; G\frac{\partial}{\partial p_\kappa}G^{-1}\; G\frac{\partial}{\partial p_\lambda}G^{-1}\; G\frac{\partial}{\partial p_\mu}G^{-1}\ . \tag{A.1}$$

With this matrix density, we have the following expressions for the topological charge N_a of the particular Fermi point with label a and for the general density of topological charge:

$$N_a = \mathrm{tr}\oint_{\Sigma_a} dS^\nu\, \mathcal{N}_\nu\ ,$$
$$\mathrm{tr}\,\frac{\partial}{\partial p_\nu}\mathcal{N}_\nu = \sum_a N_a \prod_{\mu=0}^{3} \delta\left(p_\mu - p_{\mu a}\right)\ . \tag{A.2}$$

Here, Σ_a is a three-dimensional surface around the isolated Fermi point $p_{\mu a} = (\,0, \mathbf{p}_a^{(0)}\,)$ and 'tr' stands for the trace over all indices of the Green's function, normalized by $\mathrm{tr}\,\mathbb{1} = 2\,N_{\mathrm{Weyl}}$, with N_{Weyl} the number of two-component Weyl spinors.

Equations (A.1) and (A.2) make clear that the anomalous part of the Chern–Simons vector $k_\mu = (0, \mathbf{k})$ in Eq. (12) can be written as follows:

$$k_\mu = \frac{1}{24\pi^2}\ \mathrm{tr}\int d^4p\ \ p_\mu\ \mathcal{Q}^2\ \frac{\partial}{\partial p_\nu}\mathcal{N}_\nu\ , \tag{A.3}$$

where the integral is over the whole four-dimensional energy-momentum space and $\mathcal{Q}$ is the matrix of the charges (q_a) of the fermions interacting with the $U(1)$ gauge field $\mathbf{A}$. This matrix of charges commutes with the Green's function matrix provided the $U(1)$ gauge symmetry holds. Integrating by parts and assuming the matrices $\mathcal{N}_\nu$ to vanish fast enough towards infinity, one obtains

$$k_\mu = -\frac{1}{24\pi^2}\,\mathrm{tr}\int d^4p\;\; \mathcal{Q}^2\,\mathcal{N}_\mu\;. \tag{A.4}$$

A surface integral must be added, if the matrices $\mathcal{N}_\nu$ do not vanish fast enough.

It can be shown that Eq. (A.4) is invariant under perturbations of the Green's function G which do not violate the $U(1)$ gauge symmetry and do not change the positions of the singular points. The restricted topological stability of the Fermi-point contribution to the vector $\mathbf{k}$ shows that this contribution cannot depend on space and time. The corresponding Chern–Simons-like term (1) is thus gauge invariant.

The robustness of the anomalous vector $\mathbf{k}$ under variations of the Green's function implies that it does not depend on changes of the ultraviolet cutoff. Naively, we do not expect a dependence on the infrared cutoff either. Hence, the anomalous part of $\mathbf{k}$ does not undergo radiative corrections. This suggests a type of "nonrenormalization theorem" for $\mathbf{k}$, which resembles the Adler–Bardeen theorem for the chiral anomaly.[32,33]

Similar integrals which are invariant under restricted deformations of the Green's function give rise to quantization of physical parameters in (2+1)-dimensional condensed-matter systems, examples being the Hall and spin-Hall conductivity. These parameters are expressed in terms of momentum space integrals which are invariant under deformations preserving a particular symmetry of the system (see, e.g., Sections 12.3.1 and 21.2.3 in Ref. 2 and Ref. 31). In all these cases, the Chern–Simons terms can be written as the product of coordinate-space and momentum-space topological terms. Essentially the same holds for Eqs. (1) and (A.4) in $3+1$ dimensions.

For (3+1)-dimensional condensed-matter systems, the Chern–Simons-like term (1) with vector (A.4) suggests that there may be a nonanalytic contribution to the Hall conductivity tensor σ_{H}^{mn} in bulk nonsuperconducting materials. With the definition $j^m = \sigma_{\mathrm{H}}^{mn}\,E_n$, for the electric field $\mathbf{E}$, we find:

$$\mathbf{j} = \frac{\delta S_{\mathrm{CS-like}}}{\delta \mathbf{A}} = 2\,\mathbf{k}\times\mathbf{E}\;,\quad \sigma_{\mathrm{H}}^{mn} = 2\,\epsilon^{mnl}\,k_l\;. \tag{A.5}$$

This contribution from the Fermi points comes in addition to the regular Hall conductivity of three-dimensional systems as discussed in the literature (see, e.g., Ref. 34 and references therein). The regular vector $\mathbf{k}^{\mathrm{reg}}$ in fully-gapped systems is determined by one of the vectors $\mathbf{G}$ on the reciprocal lattice in crystals or in spin/charge density waves, $\mathbf{k}^{\mathrm{reg}} = (e^2/8\pi^2\hbar)\,\mathbf{G}$; cf. Ref. 35. For the anomalous vector $\mathbf{k}$ in these periodic systems, the spatial part of the four-dimensional volume in

Eq. (A.4) includes the elementary cell of the reciprocal lattice (Brillouin zone) and 'tr' has a trace over the band indices.

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